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The contact problem for hollow and solid cylinders with stress-free faces☆

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ABSTRACT

The contact problem for hollow and solid circular cylinders with a symmetrically fitted belt and stress-free faces is considered. Homogeneous solutions corresponding to zero stresses on the cylinder faces are obtained. The generalized orthogonality of homogeneous solutions is used to satisfy the modified boundary conditions. In the final analysis the problem is reduced to a system of integral equations in the functions describing the displacement of the outer and inner surfaces of the cylinders. These functions are sought in the form of the sum of a trigonometric series and a power function with a root singularity. The ill-posed infinite systems of algebraic equations obtained as a result, are regularized by introducing small positive parameters [Ref. Kalitkin NN. Numerical Methods. Moscow: Nauka; 1978] and, after reduction, have stable regularized solutions. Since the elements of the matrices of the system are given by poorly converging numerical series, an effective method of calculating the residues of these series is developed. Formulae for the distribution function of the contact pressure and the integral characteristic are obtained. Since the first formula contains a third-order derivative of the functional series, a numerical differentiation procedure is employed when using it [Refs. Kalitkin NN. Numerical Methods. Moscow: Nauka; 1978; Danilina NI, Dubrovskaya NS, Kvasha OP et al. Numerical Methods. A Student Textbook. Moscow: Vysshaya Shkola; 1976]. Examples of the analysis of a cylindrical belt are given.

1. Formulation of the problem and the homogeneous solutions

We consider, in a cylindrical system of coordinates r, φ, z, the axisymmetric problem of the contact between a hollow elastic cylinder with radii R_0, R_1 ($0 < R_0 < R_1$) and finite length ($|z| \le 1$) with a symmetrically fitted rigid belt, having a width 2a and a base $r = R_1 - \delta(z)$, where $\delta(z)$ is a function, even in z (Fig. 1). We will assume that there are no friction forces in the belt - cylinder contact area, and the faces of the cylinder and its surface $r = R_0$ are not loaded. The boundary conditions can then be written in the form

$$\sigma_z(r, \pm 1) = \tau_{rz}(r, \pm 1) = 0, \quad R_0 \le r \le R_1 \tag{1.1}$$

$$\tau_{rz}(R_1, z) = \tau_{rz}(R_0, z) = \sigma_r(R_0, z) = 0, \quad |z| \le 1; \; u_r(R_1, z) = -\delta(z), \quad |z| \le a \tag{1.2}$$

$$\sigma_r(R_1, z) = 0, \quad a < |z| \le 1 \tag{1.3}$$

We will use the general representation of the solution of an axisymmetric problem in terms of the Love biharmonic function $\Phi(r, z)$ (Ref. 3)

$$\Delta^2\Phi = 0, \quad \Delta \equiv \partial_r^2 + r^{-1}\partial_r + \partial_z^2, \quad \partial_r \equiv \partial/\partial r, \quad \partial_z \equiv \partial/\partial z, \quad 2Gu_r = -\partial_r\partial_z\Phi$$

$$2Gu_z = [(2-2\nu)\Delta - \partial_z^2]\Phi, \quad \sigma_r = (\nu\Delta - \partial_r^2)\partial_z\Phi, \quad \sigma_z = [(2-\nu)\Delta - \partial_z^2]\partial_z\Phi$$

$$\tau_{rz} = \partial_r[(1-\nu)\Delta - \partial_z^2]\Phi, \quad \sigma_\varphi = (\nu\Delta - r^{-1}\partial_r)\partial_z\Phi \tag{1.4}$$

where G is the shear modulus and ν is Poisson's ratio.

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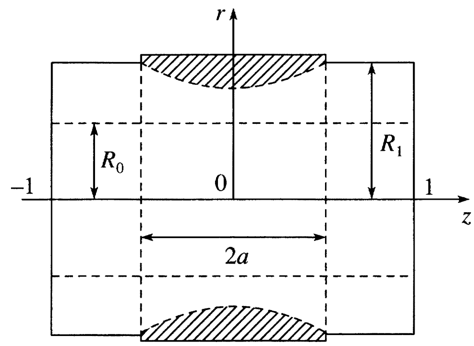

Fig. 1.

For a hollow cylinder we will seek Love's function in the form $\Phi = f^{(0)}(r)\psi(z)$. Here

$$f^{(s)}(r) = c_1 H_s^{(1)}(\gamma r) + c_2 H_s^{(2)}(\gamma r), \quad s = 0, 1$$

$H_s^{(1)}(\gamma r)$ and $H_s^{(2)}(\gamma r)$ are Hankel functions, and c_1, c_2 and γ are constants. From relations (1.4) we obtain

$$\Delta^2\Phi \equiv f^{(0)}(r)(\partial_z^2 - \gamma^2)^2\psi(z) = 0, \quad \psi(z) = C_1 \operatorname{sh}\gamma z + C_2 z \operatorname{ch}\gamma z \ (C_1, C_2 - \text{const})$$
$$2Gu_r = \gamma f^{(1)}(r)\psi'(z), \quad 2Gu_z = f^{(0)}(r)(\psi''(z) - 2\chi(z)), \quad \tau_{rz} = \gamma f^{(1)}(r)\chi(z)$$
$$\sigma_z = f^{(0)}(r)\chi^*(z), \quad \sigma_r = f^{(0)}(r)\chi'(z) - r^{-1}\gamma f^{(1)}(r)\psi'(z)$$
$$\chi(z) = \nu\psi''(z) + (1-\nu)\gamma^2\psi(z), \quad \chi^*(z) = (1-\nu)\psi'''(z) - (2-\nu)\gamma^2\psi'(z) \tag{1.5}$$

Satisfying boundary conditions (1.1), we obtain the relations

$$\chi(\pm 1) = C_1\gamma^2 \operatorname{sh}\gamma + C_2\gamma(2\nu \operatorname{sh}\gamma + \gamma \operatorname{ch}\gamma) = 0$$
$$\chi^*(\pm 1) = -C_1\gamma^3 \operatorname{ch}\gamma + C_2\gamma^2((1-2\nu)\operatorname{ch}\gamma - \gamma \operatorname{sh}\gamma) = 0$$

The non-trivial solutions in this system can be expressed in terms of the roots γ_n of the equation

$$\operatorname{sh}2\gamma_n + 2\gamma_n = 0; \quad \operatorname{Re}\gamma_n \geq 0, \quad n = 0, 1, \ldots \tag{1.6}$$

Below we give the asymptotic form of these roots and an iterational scheme for calculating them

$$2\gamma_n = \zeta_n + i\mu_n, \quad \zeta_n = \ln(2\mu_n)(1 - i/\mu_n) + O((\ln(2\mu_n)/\mu_n)^2), \quad \mu_n = 2\pi(n - 1/4)$$
$$\zeta_n^{(r+1)} = \zeta_n^{(r)} + (\mu_n - \operatorname{ch}\zeta_n^{(r)} - i\zeta_n^{(r)})/(\operatorname{sh}\zeta_n^{(r)} + i), \ \zeta_n^{(0)} = \ln(2\mu_n)(1 - i/\mu_n), \ r = 0, 1, \ldots, 5$$

Putting

$$C_1 = -(1/\operatorname{sh}\gamma_n + 2\nu\beta_n)/2, \ C_2 = \gamma_n\beta_n/2, \ \beta_n = (\gamma_n \operatorname{ch}\gamma_n)^{-1}$$

in the second equation of (1.5), we obtain the eigenfunction $\Psi_n(z)$ and the stress-strain state, corresponding to the non-zero eigenvalue γ_n ($n = 1, 2, \ldots$)

$$\Psi_n(z) = F_n'(z) - \nu\beta_n \operatorname{sh}\gamma_n z, \ \chi_n(z) = \gamma_n^2 F_n'(z), \ \chi_n^*(z) = -\gamma_n^4 F_n(z), \ \Phi_n = f_n^{(0)}(r)\Psi_n(z)$$
$$F_n(z) = \frac{1}{2}(z \operatorname{sh}\gamma_n z - \operatorname{ch}\gamma_n z \operatorname{th}\gamma_n)\beta_n = \frac{1}{\gamma_n^2}\left(F_n''(z) - \frac{\operatorname{ch}\gamma_n z}{\operatorname{ch}\gamma_n}\right), \quad \sigma_z^{(n)} = f_n^{(0)}(r)\chi_n^*(z)$$
$$\tau_{rz}^{(n)} = \gamma_n f_n^{(1)}(r)\chi_n(z), \ \sigma_r^{(n)} = f_n^{(0)}(r)\chi_n'(z) - r^{-1}\gamma_n f_n^{(1)}(r)\Psi_n'(z)$$
$$2Gu_z^{(n)} = f_n^{(0)}(r)(\Psi_n''(z) - 2\chi_n(z)), \ 2Gu_r^{(n)} = \gamma_n f_n^{(1)}(r)\Psi_n'(z)$$
$$f_n^{(s)}(r) = c_{1,n}H_s^{(1)}(\gamma_n r) + c_{2,n}H_s^{(2)}(\gamma_n r) \ \ (c_{1,n}, c_{2,n} - \text{const}) \tag{1.7}$$

The following correspond to the root $\gamma_0 = 0$

$$\Phi_0 = c_{1,0}[(2-\nu)z^3/3 - (1-\nu)zr^2/2] + c_{2,0}z\ln r, \quad \sigma_r^{(0)} = c_{1,0}(1+\nu) + c_{2,0}r^{-2}$$
$$\sigma_z^{(0)} = \tau_{rz}^{(0)} = 0, \quad 2Gu_r^{(0)} = c_{1,0}(1-\nu)r - c_{2,0}r^{-1}, \quad 2Gu_z^{(0)} = -2\nu c_{1,0}z \tag{1.8}$$

From relations (1.7) and (1.8) we obtain for $r = R_s$ ($s = 0, 1$)

$$\tau_{rz}^{(n)}(R_s, z) = f_{n,s}\chi_n(z), \quad 2Gu_r^{(n)}(R_s, z) = f_{n,s}\Psi_n'(z), \; 2Gu_r^{(0)}(R_s, z) = (1-\nu)f_{0,s}$$
$$f_{0,s} = c_{1,0}R_s - c_{2,0}R_s^{-1}/(1-\nu), \quad \sigma_r^{(n)}(R_s, z) = f_n^{(0)}(R_s)\chi_n'(z) - R_s^{-1}f_{n,s}\Psi_n'(z)$$
$$f_{n,s} = \gamma_n f_n^{(1)}(R_s), \quad 2Gu_z^{(n)}(R_s, z) = f_n^{(0)}(R_s)(\Psi_n''(z) - 2\chi_n(z)) \tag{1.9}$$
$$\sigma_r^{(0)}(R_s, z) = \sum_{h=0}^{1} f_{0,h}A_{0,s}^h, \quad \sigma_z^{(n)}(R_s, z) = f_n^{(0)}(R_s)\chi_n^*(z)$$
$$A_{0,s}^s = R_s^{-1}[1 + \nu + 2(-1)^k R_k^2/\Delta_0], \; A_{0,s}^k = 2(-1)^s R_k/\Delta_0, \; \Delta_0 = R_1^2 - R_0^2 \; (k \neq s = 0, 1)$$
$$\sigma_r(R_s, z) = \sum_{n=0}^{\infty} \sigma_r^{(n)}(R_s, z), \; u_r(R_s, z) = \sum_{n=0}^{\infty}{}' u_r^{(n)}(R_s, z), \; \tau_{rz}(R_s, z) = \sum_{n=1}^{\infty} \tau_{rz}^{(n)}(R_s, z) \tag{1.10}$$

Here and henceforth the prime on the summation sign denotes the truncated form

$$\sum_{n=0}^{\infty}{}' G_n(z) \equiv G_0(z) + 2\mathrm{Re}\left\{\sum_{n=1}^{\infty} G_n(z)\right\} \quad (\mathrm{Re}\gamma_n, \mathrm{Im}\gamma_n > 0)$$

For a solid cylinder of radius R (Fig. 1 with $R_0 = 0$, $R_1 = R$, $0 \le r \le R$) Love's function will be sought in the form $\Phi = J_0(\gamma r)\psi(z)$, while relations of the type (1.9) have the form

$$\sigma_r^{(0)}(R, z) = f_0R^{-1}(1+\nu), \; \sigma_r^{(n)}(R, z) = f_n(\gamma_n^{-1}A_n\chi_n'(z) - R^{-1}\Psi_n'(z)), \; f_n = b_n\gamma_nJ_1(\gamma_nR)$$
$$\tau_{rz}^{(n)}(R, z) = f_n\chi_n(z), \; 2Gu_r^{(0)}(R, z) = f_0(1-\nu), \; 2Gu_r^{(n)}(R, z) = f_n\Psi_n'(z), \; f_0 = b_0R$$
$$2Gu_z^{(n)}(R, z) = f_n\gamma_n^{-1}A_n(\Psi_n''(z) - 2\chi_n(z)), \; A_n = J_n(\gamma_nR)/J_1(\gamma_nR) \; (b_0, b_n - \text{const}) \tag{1.11}$$

Here $J_0(\gamma_nR)$, $J_1(\gamma_nR)$ are Bessel functions.

Applying Betti's theorem to the homogeneous solutions (1.9) and (1.11), corresponding to the two different eigenvalues γ_n and $\gamma_m (m \neq n)$, we obtain the condition for their generalized orthogonality[5]

$$\int_{-1}^{1} [F_n'(z)\beta_m \mathrm{sh}\gamma_m z + F_m'(z)\beta_n \mathrm{sh}\gamma_n z]dz = \begin{cases} 0, & m \neq n \\ -\gamma_n^{-2}, & m = n \end{cases} \tag{1.12}$$

2. Method of solution

We introduce the following notation

$$u_r(R_0, z) = u_0(z) \equiv -u(z), \quad |z| \le 1, \quad u_r(R_1, z) = u_1(z) \equiv \begin{cases} -\delta(z), & |z| \le a \\ -g(z), & a \le |z| \le 1 \end{cases} \tag{2.1}$$

Here $u(z)$ and $g(z)$ are the required functions, even in z. Then, the second boundary condition (1.2), supplemented by the first relation of (2.1), can be written in the form

$$u_r(R_s, z) = u_s(z), \quad |z| \le 1, \quad s = 0, 1 \tag{2.2}$$

Since the functional series (1.10), which determine the left-hand sides of the first condition (1.2) and conditions (1.3) and (2.2), diverge (this can be shown by an a posteriori analysis of the solution), the above boundary conditions can be replaced by the following:

$$\int_0^z\int_1^\eta \tau_{rz}(R_s, \xi)d\xi d\eta \equiv \sum_{n=1}^{\infty}{}' f_{n,s}(F_n'(z) - \beta_n \mathrm{sh}\gamma_n z) = 0, \quad |z| \le 1, \quad s = 0, 1 \tag{2.3}$$

$$2G\int_0^z u_r(R_s,\xi)d\xi \equiv f_{0,s}(1-\nu)z + {\sum_{n=1}^{\infty}}' f_{n,s}(F_n'(z) - \nu\beta_n \operatorname{sh}\gamma_n z) = 2G\int_0^z u_s(\xi)d\xi \tag{2.4}$$

$$\sigma(R_s,z) \equiv \int_1^z\int_1^\eta\int_1^t \sigma_r(R_s,\xi)d\xi d\eta dt \equiv {\sum_{h=0}^{1}}'\left\{\frac{1}{2}f_{0,h}A_{0,s}^h f(z) + \frac{1}{4}\sum_{n=1}^{\infty} f_{n,h}A_{n,s}^h\tilde{F}_n(z)\right\} +$$

$$+\frac{1}{4}c_s{\sum_{n=1}^{\infty}}' f_{n,s}\tilde{H}_n(z) = 0 \quad \text{for} \quad s = 1, \quad a \le z \le 1 \text{ and for } s = 0, \quad 0 \le z \le 1 \tag{2.5}$$

Here

$$\tilde{F}_n(z) = \frac{4}{\gamma_n}F_n'(z) + \frac{4}{\gamma_n^2}\left(\operatorname{th}\gamma_n - \frac{\operatorname{sh}\gamma_n z}{\operatorname{ch}\gamma_n}\right), \quad \tilde{H}_n(z) = \frac{1}{\gamma_n^3}\left(\frac{\operatorname{sh}\gamma_n z}{\operatorname{ch}\gamma_n} - \operatorname{th}\gamma_n\right) - \frac{z-1}{\gamma_n^2}$$

$$A_{n,s}^s = \frac{H_0^{(s+1)}(R_s\gamma_n) - H_1^{(s+1)}(R_k\gamma_n)A_{n,s}^k}{H_1^{(s+1)}(R_s\gamma_n)} - \frac{1}{R_s\gamma_n}, \quad A_{n,s}^k = (-1)^s\frac{4}{\pi i R_s\gamma_n\Delta_n}$$

$$\Delta_n = H_1^{(1)}(R_0\gamma_n)H_1^{(2)}(R_1\gamma_n) - H_1^{(1)}(R_1\gamma_n)H_1^{(2)}(R_0\gamma_n), \quad c_s = \frac{4\nu}{R_s}, \quad f(z) = \frac{(z-1)^3}{3}$$

$$n = 1, 2, \ldots; \quad k \ne s = 0, 1$$

Eqs. (2.3) and (2.4) are equivalent to the following system of relations

$${\sum_{n=1}^{\infty}}' f_{n,s}F_n'(z) = 2\theta\int_0^z u_s(\xi)d\xi - f_{0,s}z, \quad {\sum_{n=1}^{\infty}}' f_{n,s}\beta_n\operatorname{sh}\gamma_n z = 2\theta\int_0^z u_s(\xi)d\xi - f_{0,s}z, \quad |z| \le 1 \tag{2.6}$$

Here

$$f_{0,s} = 2\theta\int_0^1 u_s(\xi)d\xi, \quad \theta = \frac{G}{1-\nu} \tag{2.7}$$

Further, using the condition of generalized orthogonality (1.12), we can determine the constants $f_{n,s}$. Multiplying the first equation of (2.6) by $\beta_m\operatorname{sh}\gamma_m z$, and the second by $F'_m(z)$, and then adding and integrating with respect to z, we obtain

$$f_{n,s} = 4\theta\int_0^1 u_s(\xi)F_n''(\xi)d\xi, \quad s = 0, 1; \quad n = 1, 2, \ldots \tag{2.8}$$

Replacing the coefficients $f_{0,s}, f_{1,s}, f_{2,s}, \ldots$ in relation (2.5) by the integrals (2.7) and (2.8) and taking equalities (2.1) into account, we can write condition (2.5) in the form

$$\sigma(R_s,z) = -\theta\left\{\int_0^1 u(\xi)K_{0,s}d\xi + \int_a^1 g(\xi)K_{1,s}d\xi + \int_0^a \delta(\xi)K_{1,s}d\xi\right\} = 0 \tag{2.9}$$

for $s = 1$, $a \le z \le 1$ and for $s = 0$, $0 \le z \le 1$where

$$K_{h,s} = f_{h,s}(z) + {\sum_{n=1}^{\infty}}' F_n''(\xi)\Psi_n^{h,s}(z)$$

$$f_{h,s}(z) = A_{0,s}^h f(z), \quad \Psi_n^{s,s}(z) = A_{n,s}^s\tilde{F}_n(z) + c_s\tilde{H}_n(z), \quad \Psi_n^{k,s}(z) = A_{n,s}^k\tilde{F}_n(z), \quad k \ne s = 0, 1$$

Suppose the given function $\delta(\xi)$ and the required functions $g(\xi)$, $u(\xi)$ are defined by the series

$$\delta(\xi) = \sum_{k=0}^{\infty}\delta_k\cos a_k\xi, \quad 0 \le \xi \le a, \quad a_k = \frac{k\pi}{a}; \quad g(\xi) = \sum_{k=0}^{\infty}\delta_k g_k(\xi), \quad a \le \xi \le 1 \tag{2.10}$$

$$g_k(\xi) = X_*^{(k)} + \sum_{h=0}^{2} X_h^{(k)}(\xi-a)^{(h+1)/2} - \sum_{r=1}^{\infty} \frac{X_{r+2}^{(k)}}{l_r^2}\cos l_r(\xi-a), \quad l_r = \frac{r\pi}{l}, \quad l = 1-a$$

$$u(\xi) = \sum_{k=0}^{\infty}\delta_k u^{(k)}(\xi), \quad u^{(k)}(\xi) = \tilde{X}_0^{(k)} + \sum_{r=1}^{\infty}\frac{\tilde{X}_r^{(k)}}{b_r^2}\cos b_r\xi, - b_r = r\pi, \quad 0 \le \xi \le 1 \tag{2.11}$$

From the condition $\delta(a) = g(a)$ we obtain

$$X_*^{(k)} = (-1)^k + \sum_{r=1}^{\infty}\frac{X_{r+2}^{(k)}}{l_r^2}, \quad k = 0, 1, \ldots \tag{2.12}$$

$$g_k(\xi) = (-1)^k + \sum_{h=0}^{2} X_h^{(k)}(\xi-a)^{(h+1)/2} + \sum_{r=1}^{\infty}\frac{X_{r+2}^{(k)}}{l_r^2}(1-\cos l_r(\xi-a)), \quad a \le \xi \le 1 \tag{2.13}$$

Substituting expressions (2.10), (2.11) and (2.13) into Eqs. (2.9) and equating the coefficients of $\delta_k (k = 0, 1, \ldots)$ to zero, we obtain a system of functional equations

$$\sum_{h=0}^{\infty} X_h^{(k)}[f_h^s(z) + j_h f_{1,s}(z)] + \tilde{X}_0^{(k)} f_{0,s}(z) + \sum_{r=1}^{\infty}\tilde{X}_r^{(k)}\tilde{f}_r^s(z) = f^{k,s}(z) - \varepsilon_k f_{1,s}(z) \tag{2.14}$$

for $s = 1$, $a \le z \le 1$ and for $s = 0$, $0 \le z \le 1$ where

$$f_h^s(z) = \sum_{n=1}^{\infty}{}' Q_{h,n}\Psi_n^{1,s}(z), \quad \tilde{f}_r^s(z) = \sum_{n=1}^{\infty}{}' I_{rn}\Psi_n^{0,s}(z), \quad f^{k,s}(z) = a_k^2\sum_{n=1}^{\infty}{}'\tilde{I}_{kn}\Psi_n^{1,s}(z) \tag{2.15}$$

$$Q_{q,n} = J_n^{(q)}, \; Q_{r+2,n} = J_{rn}, \; j_q = \int_a^1 (\xi-a)^{(q+1)/2}d\xi, \quad j_{r+2} = \frac{l}{l_r^2}, \; \varepsilon_0 = 1, \; \varepsilon_r = (-1)^r l$$

$$q = 0, 1, 2; \quad r = 1, 2, \ldots; \quad s = 0, 1$$

$$J_n^{(q)} = \int_a^1 (\xi-a)^{(q+1)/2}F_n''(\xi)d\xi$$

$$J_n^{(0)} = \frac{\sqrt{l}}{4}\left[C_n(\beta_n \mathrm{sh}\gamma_n a - 4F_n'(a)) - S_n(\beta_n \mathrm{ch}\gamma_n a + 4\gamma_n F_n(a)) - \frac{\mathrm{th}\gamma_n}{\gamma_n}\right], \quad J_n^{(1)} = F_n(a)$$

$$J_n^{(2)} = \frac{3\sqrt{l}}{8\gamma_n}\left[C_n(4\gamma_n F_n(a) - \beta_n \mathrm{ch}\gamma_n a) + S_n(4F_n'(a) - 3\beta_n \mathrm{sh}\gamma_n a) + \frac{1}{\gamma_n}\right]; \quad n \le n_0$$

$$C_n = \frac{C(\gamma_n^*)}{\gamma_n^*} = \sum_{k=0}^{\infty}\frac{(\gamma_n l)^{2k}}{(2k)!(4k+1)}, \; S_n = \frac{iS(\gamma_n^*)}{\gamma_n^*} = \sum_{k=0}^{\infty}\frac{(\gamma_n l)^{2k+1}}{(2k+1)!(4k+3)}, \; \gamma_n^* = \sqrt{\frac{2\gamma_n l}{\pi i}}$$

$$J_n^{(0)} = \frac{l}{8}\sqrt{\frac{\pi}{\gamma_n}}\{\exp(\gamma_n l)[1+\tilde{\gamma}_n - \mathrm{th}\gamma_n(1-\tilde{\gamma}_n)] + i\exp(-\gamma_n l)[1-\tilde{\gamma}_n + \mathrm{th}\gamma_n(1+\tilde{\gamma}_n)]\} +$$

$$+ 2l^{3/2}\left\{\sum_{m=1}^{10}(4m-1)!!m\tilde{\gamma}_n^{2m+2} + \mathrm{th}\gamma_n\sum_{m=0}^{10}(4m+1)!!(m+1)\tilde{\gamma}_n^{2m+3}\right\}, \quad \tilde{\gamma}_n = \frac{1}{2\gamma_n l} \tag{2.16}$$

$$J_n^{(2)} = \frac{3l}{16\gamma_n}\sqrt{\frac{\pi}{\gamma_n}}\{\exp(\gamma_n l)[1-\tilde{\gamma}_n - \mathrm{th}\gamma_n(1-3\tilde{\gamma}_n)] - i\exp(-\gamma_n l)[1+\tilde{\gamma}_n + \mathrm{th}\gamma_n(1+3\tilde{\gamma}_n)]\} -$$

$$-6l^{5/2}\tilde{\gamma}_n^3\left\{\sum_{m=0}^{10}(4m+1)!!(m+1)\tilde{\gamma}_n^{2m+1} + \mathrm{th}\gamma_n\left(1+\sum_{m=1}^{10}(4m-1)!!(m+1)\tilde{\gamma}_n^{2m}\right)\right\};\ n > n_0$$

$$n_0 = \mathrm{entier}\left(\frac{7}{l}\right),\quad \int_0^a \cos a_k\xi F_n''(\xi)d\xi = (-1)^k F_n'(a) - a_k^2\tilde{I}_{kn}$$

$$J_{rn} = \frac{1}{l_r^2}\int_a^1 F_n''(\xi)(1-\cos l_r(\xi-a))d\xi = \mathrm{sh}\gamma_n\frac{(-1)^r\mathrm{sh}\gamma_n - \mathrm{sh}\gamma_n a}{(l_r^2+\gamma_n^2)^2} - \frac{F_n'(a)}{l_r^2+\gamma_n^2},\quad r = 1, 2, \ldots$$

$$I_{rn} = \frac{1}{b_r^2}\int_0^1 \cos b_r\xi F_n''(\xi)d\xi = -(-1)^r\left(\frac{\mathrm{sh}\gamma_n}{b_r^2+\gamma_n^2}\right)^2,\quad I_{kn}(-1)^k = \frac{\mathrm{sh}\gamma_n\mathrm{sh}\gamma_n a}{(a_k^2+\gamma_n^2)^2} + \frac{F_n'(a)}{a_k^2+\gamma_n^2}$$

Here $C(\gamma_n^*)$ and $S(\gamma_n^*)$ are Fresnel integrals,[4] calculated for $|\gamma_n l| \le 22$ using series (2.16).

It is easy to show (see Section 3), that the functional series (2.15) converge uniformly in the interval [0, 1], and consequently they can be integrated term by term. Multiplying Eq. (2.14) when $s=1$ by $\cos l_m(z-a)$ and when $s=0$ by $\cos b_m z$ ($m=0, 1, \ldots$) and integrating over the interval $[a, 1]$ and $[0, 1]$ respectively, we obtain two infinite systems of algebraic equations in the unknowns $X_h^{(k)}$, $\tilde{X}_h^{(k)}$ ($h=0, 1, \ldots$)

$$A\mathbf{X}^{(k)} + B\tilde{\mathbf{X}}^{(k)} = \mathbf{b}^{(k)},\ \tilde{A}\mathbf{X}^{(k)} + \tilde{B}\tilde{\mathbf{X}}^{(k)} = \tilde{\mathbf{b}}^{(k)};\ k = 0, 1, \ldots \tag{2.17}$$

Bearing in mind the integrals

$$J_r^a = \int_a^1 f(z)\cos l_r(z-a)dz = 2\frac{1-(-1)^r}{l_r^4} - \frac{l^2}{l_r^2},\quad J_0^a = -\frac{l^4}{12},\quad \varepsilon_0' = l,\quad \varepsilon_r' = 0$$

$$J_r^0 = \int_0^1 f(z)\cos b_r z dz = 2\frac{1-(-1)^r}{b_r^4} - \frac{1}{b_r^2},\quad J_0^0 = -\frac{1}{12},\quad \varepsilon_0'' = \frac{l^2}{2},\quad \varepsilon_r'' = \frac{1-(-1)^r}{l_r^2}$$

$$f_{mn}^a = \int_a^1 \tilde{F}_n(z)\cos l_m(z-a)dz = 4\varepsilon_m'\frac{\mathrm{th}\gamma_n}{\gamma_n^2} - \frac{4\gamma_n F_n(a)}{l_m^2+\gamma_n^2} - 4\frac{\mathrm{sh}\gamma_n\mathrm{ch}\gamma_n a + \gamma_n(-1)^m}{(l_m^2+\gamma_n^2)^2}$$

$$h_{mn}^a = \int_a^1 \tilde{H}_n(z)\cos l_m(z-a)dz = \varepsilon_m''\frac{1}{\gamma_n^2} - \varepsilon_m'\frac{\mathrm{th}\gamma_n}{\gamma_n^3} + \frac{(-1)^m - \mathrm{ch}\gamma_n a/\mathrm{ch}\gamma_n}{\gamma_n^2(l_m^2+\gamma_n^2)},\quad r = 1, 2, \ldots$$

$$f_{mn}^0 = \int_0^1 \tilde{F}_z(z)\cos b_m z dz,\ h_{mn}^0 = \int_0^1 \tilde{H}_n(z)\cos b_m z dz\quad (f_{mn}^0 = f_{mn}^a,\ h_{mn}^0 = h_{mn}^a \text{ for } a = 0, l = 1) \tag{2.18}$$

we obtain expressions for the elements of the matrices A, B, $\tilde{A}$, $\tilde{B}$ and the vectors $\tilde{\mathbf{b}}^{(k)}$, $\tilde{b}^{(k)}$

$$a_{m,h} = j_h A_{0,1}^1 J_m^a + \sum_{n=1}^{\infty}{}' Q_{h,n}(A_{n,1}^1 f_{mn}^a + c_1 h_{mn}^a),\quad b_{m,0} = A_{0,1}^0 J_m^a,\quad b_{m,r} = \sum_{n=1}{}' I_{rn}A_{n,1}^0 f_{mn}^a$$

$$\tilde{a}_{m,h} = j_h A_{0,0}^1 J_m^0 + \sum_{n=1}^{\infty}{}' Q_{h,n}A_{n,0}^1 f_{mn}^0,\ \tilde{b}_{m,0} = A_{0,0}^0 J_m^0,\ \tilde{b}_{m,r} = \sum_{n=1}^{\infty}{}' I_{rn}(A_{n,0}^0 f_{mn}^0 + c_0 h_{mn}^0)$$

$$b_m^{(k)} = a_k^2\sum_{n=1}^{\infty}{}'\tilde{I}_{kn}(A_{n,1}^1 f_{mn}^a + c_1 h_{mn}^a) - \varepsilon_k A_{0,1}^1 J_m^a,\quad \tilde{b}_m^{(k)} = a_k^2\sum_{n=1}^{\infty}{}'\tilde{I}_{kn}A_{n,0}^1 f_{mn}^0 - \varepsilon_k A_{0,0}^1 J_m^0$$

$$k, m, h = 0, 1, \ldots;\quad r = 1, 2, \ldots \tag{2.19}$$

Integral Eq. (2.9) are a consequence of the ill-posed problem, and hence both systems of (2.17) are ill-posed and they must be regularized by introducing small positive parameters α and $\tilde{\alpha}$.[1] The regularized systems have the form

$$(A^{\mathrm{T}}A+\alpha E)\mathbf{Y}^{(k)}+A^{\mathrm{T}}B\tilde{\mathbf{Y}}^{(k)} = A^{\mathrm{T}}\mathbf{b}^{(k)}, \quad \tilde{B}^{\mathrm{T}}\tilde{A}\mathbf{Y}^{(k)}+(\tilde{B}^{\mathrm{T}}\tilde{B}+\tilde{\alpha}E)\tilde{\mathbf{Y}}^{(k)} = \tilde{B}^{\mathrm{T}}\tilde{\mathbf{b}}^{(k)} \tag{2.20}$$

Hence, we determine the regularized solutions $\mathbf{Y}^{(k)}$, $\tilde{\mathbf{Y}}^{(k)}$ and the functions

$$g_k(\xi) = (-1)^k+\sum_{h=0}^{2}Y_h^{(k)}(\xi-a)^{(h+1)/2}+2\sum_{r=1}^{\infty}Y_{r+2}^{(k)}\left[\frac{1}{l_r}\sin\left(l_r\frac{\xi-a}{2}\right)\right]^2 \tag{2.21}$$

$$u^{(k)}(\xi) = \tilde{Y}_0^{(k)}+\sum_{r=1}^{\infty}\tilde{Y}_r^{(k)}\frac{\cos b_r\xi}{b_r^2}, \quad k = 0, 1, \dots \tag{2.22}$$

Then, from formulae (2.9) we obtain the functions $\sigma(R_s, z)$ $(s=0, 1)$, in terms of which the stresses $\sigma_r(R_s, z)=\sigma'''(R_s, z)$ are expressed. We have

$$\sigma(R_s, z) = \theta\sum_{k=0}^{\infty}\delta_k\sigma_k(R_s, z), \quad \sigma_k(R_s, z) = -\alpha_s^{(k)}f(z)-\omega^{(k)}(R_s, z)$$

$$\alpha_s^{(k)} = A_{0,s}^1\left[\varepsilon_k+\sum_{h=0}^{\infty}Y_h^{(k)}j_h\right]+A_{0,s}^0\tilde{Y}_0^{(k)}, \ \omega^{(k)}(R_s, z) = \sum_{h=0}^{\infty}Y_h^{(k)}f_h^s(z)+\sum_{r=1}^{\infty}\tilde{Y}_r^{(k)}\tilde{f}_r^s(z)-f^{k,s}(z) \tag{2.23}$$

When $n\geq$ entire $(9/R_0)$ the quantities $A_{n,s}^s$ $(s=0, 1)$ and the hyperbolic functions, for example, $\mathrm{ch}\gamma_n z$, can be calculated from the formulae

$$A_{n,s}^s = \frac{J(0,n,s)+i(-1)^s\tilde{J}(0,n,s)}{\tilde{J}(1,n,s)-i(-1)^sJ(1,n,s)}-\frac{1}{R_s\gamma_n}, \quad J(\nu,n,s) = \sum_{k=0}^{10}\frac{(-1)^k\psi(\nu,2k)}{(2R_s\gamma_n)^{2k}}$$

$$\tilde{J}(\nu,n,s) = \sum_{k=0}^{10}\frac{(-1)^k\psi(\nu,2k+1)}{(2R_s\gamma_n)^{2k+1}}, \quad \psi(\nu,k) = \frac{1}{2^{2k}k!}\prod_{m=1}^{k}[4\nu^2-(2m-1)^2], \ \psi(\nu,0) = 1$$

$$\mathrm{ch}\gamma_n z = \mathrm{ch}(\gamma(z,n)), \ \gamma(z,n) = z\zeta_n/2+2\pi i\,\mathrm{mod}[z(n/2-1/8), 1]$$

For a solid cylinder ($R_0=0$ and $R_1=R$), boundary condition (2.9) when $s=1$, the functional equation (2.14) when $s=1$ and the systems of algebraic Eqs. (2.17) and (2.20) have the form

$$\sigma(R,z) = -\theta\left\{\int_a^1 g(\xi)K(\xi,z)d\xi+\int_0^a\delta(\xi)K(\xi,z)d\xi\right\} = 0, \quad a\leq z\leq 1$$

$$\sum_{h=0}^{\infty}X_h^{(k)}[f_h(z)+j_h\tilde{f}(z)] = f^{(k)}(z)-\varepsilon_k\tilde{f}(z), \quad a\leq z\leq 1 \tag{2.24}$$

$$A\mathbf{X}^{(k)} = \mathbf{b}^{(k)}, \ (A^{\mathrm{T}}A+\alpha E)\mathbf{Y}^{(k)} = A^{\mathrm{T}}\mathbf{b}^{(k)}, \ k = 0, 1, \dots \tag{2.25}$$

where

$$K(\xi,z) = \tilde{f}(z)+\sum_{n=1}^{\infty}{}'F_n''(\xi)\tilde{\Psi}_n(z), \quad \tilde{f}(z) = \frac{1+\nu}{R}f(z), \quad \tilde{\Psi}_n(z) = B_n\tilde{F}_n(z)+c_1\tilde{H}_n(z)$$

$$B_n = A_n-\frac{1}{\gamma_nR}, \quad f_h(z) = \sum_{n=1}^{\infty}{}'Q_{h,n}\tilde{\Psi}_n(z), \quad f^{(k)}(z) = a_k^2\sum_{n=1}^{\infty}{}'\tilde{I}_{kn}\tilde{\Psi}_n(z)$$

$$a_{m,h} = J_m^a\frac{1+\nu}{R}j_h+\sum_{n=1}^{\infty}{}'Q_{h,n}\tilde{J}_{mn}, \quad b_m^{(k)} = a_k^2\sum_{n=1}^{\infty}{}'\tilde{I}_{kn}\tilde{J}_{mn}-\varepsilon_kJ_m^a\frac{1+\nu}{R}$$

$$\tilde{J}_{mn} = B_nf_{mn}^a+c_1h_{mn}^a; \quad m, h = 0, 1, \dots$$

3. Calculation of the residues of the numerical and functional series

Series (2.15) and (2.19) can be represented in the form

$$\sum_{n=1}^{p}{}' J_n^{(h)}\Psi_n^{1,1}(z) + R_p^{(h)}(z), \quad \sum_{=1}^{p}{}' J_n^{(h)}(A_{n,1}^1 f_{mn}^a + c_1 h_{mn}^a) + R_p^h(m), \quad h = 0, 1, 2 \text{ etc.}$$

Here $R_p^{(h)}(z)$, $R_p^h(m)$ are the residues, beginning with the $(p+1)$-th term, $p \geq 4000$.

If we introduce the notation

$$\lambda_n = (-4\gamma_n)^{-1}, \quad z(\theta) = \exp(i\pi\theta), \quad x_r = r/(lp), \quad l_r/\gamma_n = -4\pi p x_r \lambda_n$$

and solve Eq. (1.6) for $\exp(2\gamma_n)$, then, raising $\exp(2\gamma_n)$ to the power a, we obtain

$$\exp(2\gamma_n a) = z^n(2a)\lambda_n^{-a}[1/2 + \sqrt{1+4\lambda_n^2}/2]^a = z^n(2a)\lambda_n^{-a}[1 + a\lambda_n^2 + a(a-3)\lambda_n^4/2 + \ldots]$$

Taking this formula into account for large n and small l_n/γ_n, the integrals $J_n^{(h)}, J_{zn}, \ldots, h_{mn}^0$, the quantities $A_{n,s}^s$ and the functions $\tilde{F}_n(z)$, $\tilde{H}_n(z)$ can be expanded in series in powers of λ_n

$$\begin{aligned}
J_n^{(0)} &= \sqrt{\pi}\lambda_n[iq_{0,n}(a) + q_{0,n}(-a)] - 16l^{-3/2}[\lambda_n^3 - (2 + 6/l)\lambda_n^4 + \ldots] \\
J_n^{(1)} &= 2\lambda_n^{3/2}[q_{1,n}(-a) + q_{1,n}(a)], \quad \tilde{F}_n(z) = 8\lambda_n^{3/2}[\tilde{q}_n(-z) - \tilde{q}_n(z)] + 64\lambda_n^2 \operatorname{th}\gamma_n \\
\tilde{H}_n(z) &= 64\lambda_n^{7/2}[q_n^0(z) - q_n^0(-z)] + 64\lambda_n^3 \operatorname{th}\gamma_n - 16\lambda_n^2(z-1), \quad \operatorname{th}\gamma_n = 1 - 2\lambda_n + 2\lambda_n^2 - \ldots \\
A_{n,s}^s &= i(-1)^s + 2R_s^{-1}\lambda_n - 6i(-1)^s R_s^{-2}\lambda_n^2 + \ldots \\
q_{0,n}(a) &= z^n(1-a)\lambda_n^{a/2}[(1+a)/2 - \lambda_n(1+a)/2 + \lambda_n^2(5-a^2)/4 + \ldots] \\
q_{1,n}(a) &= z^n(1-a)\lambda_n^{a/2}[1 + a + \lambda_n(1-a) + \lambda_n^2(1-a^2)/2 + \ldots] \\
\tilde{q}_n(z) &= z^n(1-z)\lambda_n^{z/2}[1 + z - \lambda_n(7+z) + \lambda_n^2(17-z^2)/2 + \ldots] \\
q_n^0(z) &= z^n(1-z)\lambda_n^{z/2}[1 - \lambda_n + \lambda_n^2(1-z)/2 + \ldots] \quad \text{etc.}
\end{aligned} \tag{3.1}$$

Multiplying the expansions by the corresponding quantities from formulae (3.1) and dropping terms of higher order of smallness than λ_n^2, we obtain, with the accuracy indicated, expressions for the n-th terms of the residues

$$\begin{aligned}
&J_n^{(h)}\Psi_n^{1,1}(z) = Q_{h,n}^*, \quad J_n^{(h)}(A_{n,1}^1 f_{mn}^a + c_1 h_{mn}^a) = \\
&= \sum_{j=0}^{3} x_m^{2j}[Q_{h,n}^j + (-1)^m Q_{h+3,n}^j] + \varepsilon_m' Q_{h+3,n}^* + \varepsilon_m'' Q_{h+6,n}^*, \quad h = 0, 1, 2 \text{ etc.}
\end{aligned} \tag{3.2}$$

The expressions $Q_{h,n}^*, Q_{k,n}^j, \ldots$ are sums of a finite number of terms of the form $\lambda_n^s A$ or $z^n(\theta)\lambda_n^s A$. In view of the length of these expressions we will only give below the principal part of the most important of these

$$\begin{aligned}
Q_{0,n}^* &= 4\sqrt{\pi}\{iQ_{0,n}(a,-z) - iQ_{0,n}(a,z) + Q_{0,n}(-a,-z) - Q_{0,n}(-a,z) + \\
&+ 8[i\tilde{Q}_{0,n}(a) + \tilde{Q}_{0,n}(-a)] + \ldots\} \\
Q_{0,n}(a,z) &= z^n(-a-z)\lambda_n^{(5+a+z)/2} A_{n,1}^1[(1+a)(1+z) - 2\lambda_n(1+a)(4+z) + \ldots] \\
\tilde{Q}_{0,n}(a) &= z^n(1-a)\lambda_n^{(6+a)/2} A_n^*[1 + a - \lambda_n(1+a) + \ldots] \\
A_n^* &= (A_{n,1}^1 + \lambda_n c_1)\operatorname{th}\gamma_n - c_1(z-1)/4 = -i - c_1(z-1)/4 + \lambda_n(2i + 2R_1^{-1} + c_1) + \ldots
\end{aligned} \tag{3.3}$$

Further, we use the following summation formulae and theorem[5,6]

$$\begin{aligned}
J(s,\theta) &= \sum_{n=p+1}^{\infty} z^n(\theta)\lambda_n^s = \\
&= \frac{(1+\theta_0)z^{p+1}(\theta)}{(-4\pi i\upsilon)^s}\left\{1 - \frac{s\Theta}{\upsilon} + \frac{s[(s+1)(\Theta^2 + \theta_0^2 + \theta_0) + i\Theta/\pi]}{2\upsilon^2} + O(\upsilon^{-3})\right\}, \quad \upsilon = p + \frac{3}{4}
\end{aligned} \tag{3.4}$$

$$\Theta = \theta_0 + \tilde{\upsilon}, \quad \tilde{\upsilon} = \frac{\ln(4\pi\upsilon)}{2\pi i}, \quad \theta_0 = \frac{1}{2}\left(-1 + i\operatorname{ctg}\frac{\pi\theta}{2}\right), \quad s > 0, \quad 0 < |\theta| < 2$$

$$J(s, 0) = \sum_{n=p+1}^{\infty} \lambda_n^s = (-4\pi i\upsilon)^{-s}\left[\frac{\upsilon}{s-1} - B_1(\tilde{\upsilon}) + \frac{i}{2\pi s} + \frac{sB_2(\tilde{\upsilon})}{2\upsilon} + O(\upsilon^{-2})\right], \quad s > 1 \tag{3.5}$$

where $B_1(\tilde{\upsilon}) = \tilde{\upsilon} - 1/2,\ B_2(\tilde{\upsilon}) = \tilde{\upsilon}^2 - \tilde{\upsilon} + 1/6$ are Bernoulli polynomials.

Theorem. *Suppose the following conditions are satisfied*

$$s(M) > 0, \quad 0 < |\theta(M)| < 2, \quad M \in D$$

Then, the functional series (3.4) converges uniformly in D. If

$$s(M) > 1, \quad M \in D_0$$

the series converges uniformly in D_0 for any values of $\theta(M)$.

Having the partial sums (3.2) and (3.3) and formulae (3.5), we can investigate the convergence of the series (2.15) and (2.19) and calculate the residues of these series $R_p^{(h)}(z), R_p^h(m), \ldots$.[5] In particular, using relations (3.3), we calculate the residue

$$R_p^{(0)}(z) = \sum_{n=p+1}^{\infty}{}' Q'^{*}_{0,n} = 8\sqrt{\pi}\operatorname{Re}\{iE_0(a, -z) - iE_0(a, z) + E_0(-a, -z) - E_0(-a, z) +$$

$$+ 8[i\tilde{E}_0(a) + \tilde{E}_0(-a)] + \ldots\}$$

$$E_0(a, z) = -i(1+a)(1+z)J\left(\frac{5+a+z}{2}, -a-z\right) +$$

$$+ 2(1+a)\left[\frac{1+z}{R_1} + i(4+z)\right]J\left(\frac{7+a+z}{2}, -a-z\right) + \ldots$$

$$\tilde{E}_0(a) = \left(c_1\frac{1-z}{4} - i\right)(1+a)J\left(\frac{6+a}{2}, 1-a\right) + \left(\frac{2}{R_1} + 3i + c_1\frac{z+3}{4}\right)(1+a)J\left(\frac{8+a}{2}, 1-a\right) + \ldots$$

Note that, in integral Eqs. (2.9) and (2.24), the kernels $K_{h,s}(\xi, s)$ $(h, s = 0, 1, K(\xi, z)$ are continuous and bounded in the region $D\{\xi, z \in [0, 1]\}$, in which case the kernel $K_{h,h}(\xi, z)$, and $K(\xi, z)$ in the band $D^*\{|\xi - z| \to 0\}$ have a singularity of the type $(\zeta - z)\ln|\xi - z|$.[5]

The accuracy of the calculation of the residue R_p was monitored using the quantity ε_p. Thus, when checking the residues $R_p^{(0)}(0.5), R_p^{(0)}(0)$ $(a = 0.25, R_1 = 0.5, R_0 = 0.1, \nu = 0.3, p = 4000)$ the following values were obtained:

$$R_p^{(0)}(0.5) = 1.96 \cdot 10^{-12}, \quad \varepsilon_p = 6 \cdot 10^{-20}; \quad R_p^{(0)}(0) = 9.96 \cdot 10^{-13}, \quad \varepsilon_p = 10^{-21}$$

$$R_p = \sum_{n=p+1}^{\infty} a_n, \quad r = R_{4000} - R_{6000}, \quad \tilde{R} = \sum_{n=4001}^{6000} a_n, \quad \varepsilon_p = |r - \tilde{R}|$$

4. Determination of the contact pressure

We will present examples of the calculation of a cylindrical belt $\delta(z) \equiv \delta_0$, $k = 0$; $a = 0.25$, $R_1 = R = 0.5$ for the following versions: 1) $R_0 = 0.1$, 2) $R_0 = 0.2$ and 3) $R_0 = 0$ (a solid cylinder). The infinite systems (2.20) and (2.25) in the unknowns $Y_s^{(0)}, \tilde{Y}_s^{(0)}$ $(s = 0, 1, \ldots)$ (the zero superscript on the quantities $Y_s^{(0)}, u^{(0)}, \ldots$ is henceforth omitted) were truncated and solved for several values of α and $\tilde{\alpha}$. For each version we chose its own set of least values of the regularization parameters (the values of the pair of parameters $(\alpha, \tilde{\alpha})$ and the single parameter α were as follows: $(8 \times 10^{-19}, 6 \times 10^{-18})$, $(3 \times 10^{-19}, 5 \times 10^{-18})$ and 5×10^{-19} for Versions 1, 2 and 3 respectively), for which we have already observed considerable amplitudes of the oscillations of the regularized solutions $Y_s, \tilde{Y}_s$ $(s = 0, \ldots, 80)$, but the discrepancy was fairly small:

$$|\sigma_0(R, z)| < 10^{-8}, \quad |\sigma_0(R_1, z)| < 7 \cdot 10^{-9}, \quad a < z \le 1;$$

$$|\sigma_0(R_0, z)| < 6 \cdot 10^{-9}, \quad 0 \le z < 1$$

In Table 1 we show values of $Y_s \times 10^5$ and $\tilde{Y}_s \times 10^5$.

In Fig. 2 we show graphs of the functions $g_0(z)$ and $u^0(z) \equiv u(z)$, obtained from formulae (2.21) and (2.22), where the number on the curve corresponds to the number of the version.

Table 1

s	$Y_s \cdot 10^5$			$\tilde{Y}_s \cdot 10^5$	
	Versions				
	1	2	3	1	2
0	−246458	−204089	−263413	−17320	−30048
1	57574	−4490	81124	−273980	−473484
2	53772	79614	46640	−332328	−562318
3	273826	202414	298161	−78428	−81653
75	−407	−305	−692	−4741	−6658
76	57	144	101	4534	6427
77	397	−276	−690	−4324	−6072
78	5	128	11	4142	5878
79	−410	−264	−712	−3954	−5549
80	−50	103	−102	3797	5407

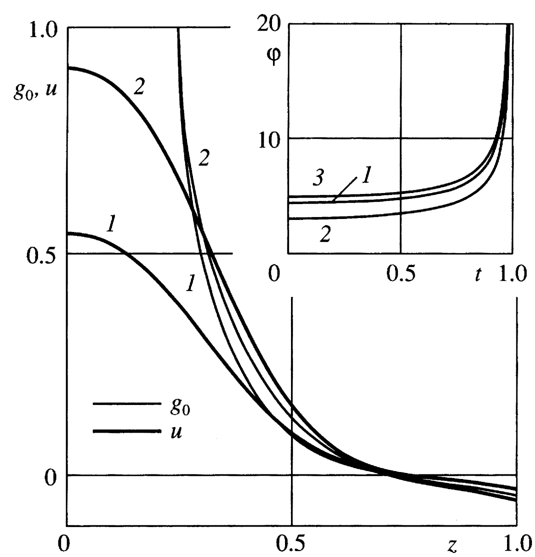

Fig. 2.

In order to obtain the contact pressure $q(z) = -\sigma(R_1, z)$ ($|z| \le a$) we use relations (2.23) with $k = 0$

$$\sigma(R_1, z) = \theta\delta_0\sigma_0(R_1, z), \; \sigma_0(R_1, z) = -\alpha_1 f(z) - \omega(R_1, z), \; \sigma_r(R_1, z) = \theta\delta_0\sigma_0'''(R_1, z)$$

$$\alpha_1 = A_{0,1}^1\left(1 + \sum_{h=0}^{80} Y_h j_h\right) + A_{0,1}^0 \tilde{Y}_0, \quad \omega(R_1, z) = \sum_{h=0}^{80} Y_h f_h^1(z) + \sum_{r=1}^{80} \tilde{Y}_r \tilde{f}_r^1(z)$$

In the case of a solid cylinder we have

$$\sigma_r(R, z) = \theta\delta_0\sigma_0'''(R, z), \quad \sigma_0(R, z) = -\alpha_0\tilde{f}(z) - \omega(R, z), \quad \alpha_0 = 1 + \sum_{s=0}^{80} Y_s j_s,$$

$$\omega(R, z) = \sum_{s=0}^{80} Y_s f_s(z)$$

Hence it follows that the dimensionless distribution functions of the contact pressure $\tilde{\varphi}(z)$ and the integral characteristic N_0 are given by the expressions

$$\tilde{\varphi}(z) = q(z)(\theta\delta_0)^{-1} = -\sigma_0'''(R_1, z) = 2\alpha_1 + \omega'''(R_1, z)$$

$$aN_0 = -2\int_0^a \sigma_0'''(R_1, z)dz = -2\sigma_0''(R_1, a) + 4\alpha_1 - 2\sum_{h=0}^{80} Y_h(f_h^1(0))'' - 2\sum_{r=1}^{80} \tilde{Y}_r(\tilde{f}_r^1(0))'' \tag{4.1}$$

Table 2

k	$\varphi(t_k)$		
	Versions		
	1	2	3
0	4.396	3.008	4.937
1	4.424	3.045	4.962
2	4.521	3.167	5.049
3	4.735	3.416	5.248
4	5.210	3.921	5.712
5	6.562	5.213	7.092

Table 3

Quantities	Versions			
	1	2	3	4
χ_1	3.026	2.297	3.312	3.420
χ_2	1.099	0.752	1.234	1.290
χ_3	0.853	0.713	0.912	0.896
$u(0)$	0.5344	0.9069	–	–
$u(1)$	−0.0429	−0.0683	–	–
$g_0(1)$	−0.0426	−0.0578	−0.0320	–

Also, for a solid cylinder

$$\tilde{\varphi}(z) = 2(1+\nu)R^{-1}\alpha_0 + \omega'''(R, z),$$

$$aN_0 = -2\sigma_0''(R, a) + 4(1+\nu)R^{-1}\alpha_0 - 2\sum_{s=0}^{80} Y_s f_s''(0) \tag{4.2}$$

Taking into account the equalities

$$\sigma_0''(R_1, a) = \sigma_0''(R, a) = (f_h^1(0))'' = (\tilde{f}_r^1(0))'' = f_s''(0) = 0$$

we obtain formulae for the integral characteristics of hollow and solid cylinders

$$N_0 = 4a^{-1}\alpha_1, \quad N_0 = 4(1+\nu)(aR)^{-1}\alpha_0$$

Further, we carry out numerical differentiation of the functions specified on a uniform grid with step $h = z_{k+1} - z_k$ (z_k are nodes of the grid). To calculate the third-order derivative $\omega'''(R_1, z)$ at the central node $z = z_0$ with respect to the seven nodes $z_k = z_0 + kh$ ($k = -3, \ldots, 3$; $0.0005 \le h \le 0.001$), a formula of increased accuracy[5] is used, namely

$$\omega'''(R_1, z_0) = h^{-3}\left(\frac{1}{8}\omega_{-3} - \omega_{-2} + \frac{13}{8}\omega_{-1} - \frac{13}{8}\omega_1 + \omega_2 - \frac{1}{8}\omega_3\right) + O(h^4), \ \omega_k = \omega(R_1, z_k)$$

In Table 2 we show the values of the function $\varphi(t) \equiv \tilde{\varphi}(at)$ $\left(t = \frac{z}{a}\right)$ for $t = t_k = k/6$, while in Table 3 we show values of the quantities ($\nu = 0.3$)

$$\chi_1 = aN_0, \quad \chi_2 = a\varphi(0), \quad \chi_3 = a\lim\varphi(t)\sqrt{1-t^2} \ (t \to 1)$$

and the functions $u(z)$ and $g_0(z)$ for $z = 0$ and $z = 1$.

Comparing the values of χ_r ($r = 1, 2, 3$) for solid and hollow cylinders of finite dimensions with the corresponding values of χ_r for an infinite solid cylinder (Version 4, see Ref. 3, p. 97), we see that they differ by less than 4.3% (Version 3), 15% (Version 1) and 42% (Version 2).

In the upper right-hand part of Fig. 2 we show graphs of the function $\varphi(t)$ obtained using formulae (4.1) and (4.2). In order to explain these graphs, we separate out the root singularity in the contact stress, using the following representation, for example, the functions $\sigma_r(R, z)$ (Version 3) in the neighbourhood of the point $z = a$

$$\sigma_r(R, z) = -\theta\delta_0(a-z)^{-1/2}L_2(z), \quad 0 \le a - z \le 3\tilde{h}; \quad \sigma_r(R, z) = 0, \quad z > a \tag{4.3}$$

Here $L_2(z) = a_0 + a_1(a-z) + a_2(a-z)^2$ is the interpolation polynomial for the function $y(z) = \tilde{\varphi}(z)\sqrt{a-z}$, specified at the interpolation nodes

$$z_k = a - k\tilde{h}, \quad k = 1, 2, 3; \quad \tilde{h} = 0.003$$

Calculating the values of $y(z_k)$ for $h = 0.0006$ and then a_k, we obtain

$$\tilde{\varphi}(z) = \frac{1}{\sqrt{a-z}}[a_0 + a_1(a-z) + a_2(a-z)^2],\ 0 \le a - z \le 0.009$$

$$a_0 = 1.2845, \quad a_1 = 1.618, \quad a_2 = 104.9$$

Note the good agreement between the quantities $a_0/\sqrt{2} = 0.908$ and $\chi_3 = 0.912$ (see Table 3).

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